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Justification of Geometrical Optics for Non-convex Obstacles*

HANS-DIETER ALBER

Institut für Angewandte Mathematic der Universität Bonn, Wegelerstrasse 10, 5300 Bonn 1, West Germany

Submitted by Cathleen Morawetz

1. INTRODUCTION AND STATEMENT OF RESULTS

Using new results about the propagation of singularities, we shall show that it suffices to approximate the exact solution of the Helmholtz equation locally for justification of geometrical optics. The methods which we describe allow us to show that the geometrical-optics approximation converges to the exact solution of the Helmholtz equation for a large class of obstacles. Apart from some technical assumptions concerning the behaviour of the reflected rays, we only need that solutions of the wave equation decay exponentially outside of the obstacle. This is true, if rays are not trapped, cf. [8]. In this paper, however, we shall not treat the most general case, but approximate the solution of the Helmholtz equation only at points x_0 satisfying the additional conditions (A2) and (A3) given below. A more general case will be considered in a sequel to this paper.

Let B be a bounded open set with C^∞ boundary in $\mathbb{R}^3$, and $\Omega = \mathbb{R}^3 - B$. The solution of

$$\Delta u + k^2 u = 0, \tag{1.1}$$

$$u|_{\partial B} = \mu, \qquad \mu \in C(\partial B), \tag{1.2}$$

$$\partial_r u = iku + o(r^{-1}), \qquad r = |x| \to \infty, \tag{1.3}$$

will be approximated for $k \to \infty$. We shall calculate the contribution of the values of μ in a neighborhood of $x_B \in \partial B$ to the value of u at $x_0 \in \Omega$ under the following assumptions.

Let S be a given ball containing B.

(A1) The local energy in $S - B$ of a solution of

$$(\partial_t^2 - \Delta)u = 0 \tag{1.4}$$

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goes to zero as $t \to \infty$, uniformly with respect to all solutions of (1.4) of unit energy which vanish outside S at time $t = 0$.

For x_0 we assume:

(A2) None of the rays starting at x_0 and reflected at ∂B as usual has more than one point in common with ∂B.

(A3) Every ray starting at x_0 is at most tangent to ∂B, to finite order.

It will be shown that condition (A1) implies exponential decay of the energy in bounded subsets of Ω, since we only consider three-dimensional space. We need assumption (A3), since we use the results in [6], which have been proved under this condition. Assumption (A2), finally, will be weakened in the sequential to this paper mentioned above. Rays involving finitely many reflections can be allowed, and it is only for simplicity in notation and proof that we state this assumption.

In the investigation of this problem points of contact of grazing rays must be considered separately from the other points of ∂B. The latter will be called regular points. First we shall study the regular points. We shall approximate the normal derivative of the Green's function $G(x, y, k)$ of problem (1.1)–(1.3) at such a regular point $x_B \in \partial B$. To this end we note that the Fourier transform

$$R(x, y, t) = (2\pi)^{-1} \int G(x, y, k)\, e^{-ikt}\, dk$$

of G is the Riemann function to the wave equation. R satisfies

$$(\partial_t^2 - \Delta)R = \delta(x - y)\, \delta(t), \tag{1.5}$$

$$R\,|_{x \in \partial B} = 0, \tag{1.6}$$

$$R = 0 \qquad \text{for} \quad t < 0. \tag{1.7}$$

As in [1] we shall approximate R by a progressing wave expansion. The Fourier transform of this progressing wave expansion yields the approximation of G. This method is superior to the direct approximation of G, since new results about the propagation of singularities (cf. [2, 3, 5, 6, 9]) allow us to show that it is sufficient to approximate R and therefore also G locally.

Along the ray starting at x_0 and reflected at x_B, $R(x, x_0, t)$ will be approximated by

$$R_\infty(x, x_0, t) = (4\pi\,|x - x_0|)^{-1}\delta(t - |x - x_0|) + \sum_{m=-1}^{\infty} \psi(\lambda_m(t - \Phi(x))\, H_m(t - \Phi(x))\, p_m(x), \tag{1.8}$$

where $H_{-1}(r) = \delta(r)$, and for $m \geqslant 0$,

$$H_m(r) = 1/m!\, r^m H(r) = 1/m!\, r^m, \qquad r \geqslant 0,$$
$$= 0, \qquad r < 0.$$

$\psi(s) \in C^\infty(\mathbb{R})$ is equal to 1 in a neighborhood of 0, and vanishes for $|s| > 1$. $\lambda_m > 0$ tends to infinity sufficiently fast for $m \to \infty$. Φ and p_m are solutions of differential equations from geometrical optics:

$$(\nabla\Phi)^2 = 1 \qquad \text{(eikonal equation)} \tag{1.9}$$

$$2\nabla\Phi \cdot \nabla p_m + \Delta\Phi p_m = \Delta p_{m-1},\ p_{-2} = 0 \qquad \text{(transport equations).} \tag{1.10}$$

For Φ we choose the function defined by $\Phi(x) =$ (distance between x_0 and x, measured along the ray starting at x_0, passing over x, and reflected between x_0 and x). It is well known that this function is a solution of (1.9) in a sufficiently small neighborhood of ∂B. We require that R_∞ vanishes at ∂B. At $x \in \partial B$, p_m therefore has to satisfy

$$p_m(x) = -(4\pi\,|x - x_0|)^{-1}, \qquad m = -1,$$
$$= 0, \qquad m \geqslant 0.$$

These relations are initial conditions for p_m. Observe that R_∞ coincides for $t < d = \operatorname{dist}(x_0, B)$ with the exact solution

$$R(x, x_0, t) = \tilde{R}(x, x_0, t) = (4\pi\,|x - x_0|)^{-1}\delta(t - |x - x_0|), \tag{1.11}$$

$\tilde{R}$ being the free space solution of (1.5) and (1.7). It will be shown that R_∞ is a parametrix for the wave operator.

Our result for regular points is:

THEOREM 1.1. *Let $x_B \in \partial B$.*

(i) *If none of the rays starting at x_0 passes over x_B, then $\partial_n G(x_B, x_0, k)$ decreases faster than any power of k. ∂_n denotes differentiation with respect to the exterior normal of ∂B.*

(ii) *If there is a ray passing over x_B, which is not tangential to ∂B at x_B, then*

$$\partial_n G(x_B, x_0, k) = e^{ik|x_B - x_0|}\left(ik\,\frac{(x_B - x_0)\cdot n}{2\pi\,|x_B - x_0|^2} - \frac{(x_B - x_0)\cdot n}{4\pi\,|x_B - x_0|^3}\right.$$
$$\left. + \sum_{m=-1}^{l} (-ik)^{-m-1}\partial_n p_m(x_B)\right) + O(k^{-l-2}), \quad k \to \infty. \tag{1.12}$$

The limit is uniform with respect to x_B varying in any compact set of regular points.

It is well known that the Green's function cannot be approximated at points of contact of grazing rays using such a simple Ansatz. Here we shall prove some results which allow us to treat these points in some cases. First, we show that the asymptotic behaviour of the Green's function depends at $x_B \in \partial B$ only on the shape of the boundary in an arbitrary small neighborhood of x_B. This allows us to reduce the problem of approximating the Green's function of complicated obstacles to the approximation of the Green's function of simple obstacles. For example, if B is strictly convex in a neighborhood of x_B, then it follows from our result, that the function constructed by Ludwig in [4] is approximate to G, since it has been proved in [8], that this is the case for strictly convex obstacles.

To formulate the result, assume that B_1 and B_2 are obstacles and that x_0 is a point in the common exterior such that (A1)–(A3) are satisfied for B_1 and B_2. Assume moreover that the boundaries ∂B_1 and ∂B_2 coincide in a neighborhood of $x_B \in \partial B_1 \cap \partial B_2$, and that the straight line between x_0 and x_B also belongs to the common exterior. Let G_1 and G_2 denote the Green's functions for B_1 and B_2. Then we have:

THEOREM 1.2. *There is a relatively open neighbourhood $U \subseteq \bar{\Omega}$ of x_B with*

$$|D_x^\alpha G_1(x, x_0, k) - D_x^\alpha G_2(x, x_0, k)| \to 0$$

for $k \to \infty$, faster than any power of k, and uniformly with respect to $x \in U$.

Using this result we can show that the contribution to $u(x_0)$ of the values of μ in a neighborhood of x_B can be neglected in some cases. Let U be a neighborhood in $\mathbb{R}^3$ of $\operatorname{supp} \mu \subseteq \partial B$, and assume that ∂B coincides in U with the boundary of a strictly convex obstacle B_1. Let x_0 be a point in the common exterior such that (A1)–(A3) are satisfied for B and B_1, and assume that $x \in \operatorname{supp} \mu$ can be connected to x_0 by a straight line in $\mathbb{R}_3 - \bar{B}$ if and only if this straight line lies in $\mathbb{R}^3 - \bar{B}_1$. Theorem 1.1(i) and Theorem 1.2 now imply for the Green's functions

$$|\partial_n G(x, x_0 k) - \partial_n G_1(x, x_0, k)| \to 0,$$

uniformly for $x \in \operatorname{supp} \mu$, hence $|u(x_0, k) - u_1(x_0, k)| \to 0$, for $k \to \infty$, faster than any power of k. Here u_1 denotes the solution of (1.1)–(1.3) with B replaced by B_1. It therefore suffices to determine $u_1(x_0, k)$. We consider the Fourier transform v of u_1. It will be seen that v satisfies

$$(\partial_t^2 - \Delta)v = 0 \qquad \text{in } \Omega, \tag{1.13}$$

$$v\,|_{x\in\partial B} = \mu(x)\,\delta(t), \tag{1.14}$$

$$v = 0 \qquad \text{for} \quad t < 0. \tag{1.15}$$

In [9] it is shown (Theorem 1.3) that for strictly convex obstacles $WF(v)$ is contained in the bicharacteristic strips of the operator $\partial_t^2 - \Delta$ passing over the elements of $WF(\mu(x)\,\delta(t)) \subseteq T^*(\partial B_1 \times \mathbb{R})$. Let $M(\mu)$ be the projection to the (x_1, x_2, x_3)-space of these bicharacteristic strips passing over $WF(\mu(x)\,\delta(t))$. Using this result we obtain:

LEMMA 1.3. *Let $x_0 \notin M(\mu)$. Then $u_1(x_0, k) \to 0$, hence also $u(x_0, k) \to 0$ for $k \to \infty$, faster than any power of k.*

Suppose now that ∂B coincides in a neighborhood of all the points of contact of the grazing rays starting at x_0 with the boundary of a strictly convex obstacle B_1, and that x_0, B, and B_1 satisfy the assumption made before Lemma 1.3. Let V_1, $V_2 \subseteq \partial B$ be an open covering of ∂B, where $V_2 \subseteq \partial B \cap \partial B_1$ is a neighborhood of the points of contact of the grazing rays starting at x_0. We choose V_2 so small, that none of the normals $n(x)$ at $x \in \bar{V}_2$ is parallel to $x - x_0$. Let χ_1, $\chi_2 \in C^\infty(\partial B)$ with $\operatorname{supp} \chi_1 \subseteq V_1$ and $\operatorname{supp} \chi_2 \subseteq V_2$ be a partition of unity. Theorem 1.1 and Lemma 1.3 imply

COROLLARY 1.4. *Let $\mu \in C(\partial B)$, and assume that μ is infinitely differentiable in V_2. Then we have for the solution u of* (1.1)–(1.3)

$$u(x_0) = \int_{\partial B} \mu(x)\,\chi_1(x)\,e^{ik|x-x_0|}\left(ik\,\frac{(x-x_0)\cdot n}{2\pi\,|x-x_0|^2} - \frac{(x-x_0)\cdot n}{4\pi\,|x-x_0|^3}\right.$$
$$\left. + \sum_{m=-1}^{l} (-ik)^{-m-1}\,\partial_n p_m(x)\right) dS_x + O(k^{-l-2}). \tag{1.16}$$

This can be seen as follows: Let $\mu_1 = \chi_1 \mu$ and $\mu_2 = \chi_2 \mu$, hence $\mu = \mu_1 + \mu_2$. $M(\mu_2)$ is contained in the set of all points $x + \gamma n(x)$ with $x \in V_2$ and $\gamma \geqslant 0$, since μ_2 is infinitely differentiable and $\operatorname{supp} \mu_2 \subseteq V_2$. Thus we have $x_0 \notin M(\mu_2)$, by definition of V_2, hence, by Lemma 1.3, the contribution of μ_2 to $u(x_0)$ decreases faster than any power of k. Equation (1.16) therefore follows from (1.12).

Assume that μ is l-times differentiable. In this case the limit of the integral in (1.16) can be calculated by the method of stationary phase, leading to the usual geometrical optics approximation of u. It is readily seen that the essential contribution to $u(x_0)$ comes from the values of μ in a neighborhood of the points x where $n(x)$ is parallel to $x - x_0$. However, if μ is not smooth, then also the values of μ at other points of ∂B can yield essential contributions to $u(x_0)$.

2. Proof of Theorem 1.1

The proof consists of three parts. In the first part we define more precisely what we mean by a solution of (1.5)–(1.7). Then we show that a unique solution exists.

Let $\mathscr{S}_0$ denote the space of all functions $\phi(x, t) \in C^\infty(\bar{\Omega} \times \mathbb{R})$ with $\Delta_x^m \phi(x, t) = 0$ for $x \in \partial B$, $m = 0, 1, 2, \ldots$, and

$$\sup_{z \in \bar{\Omega} \times R} |z^\beta D^\alpha \phi(z)| < \infty \tag{2.1}$$

for all multi-indices α, β. We supply $\mathscr{S}_0$ with the topology induced by these seminorms. By $\mathscr{S}_0'$ we denote the dual space. For $u \in \mathscr{S}_0'$ we define Δu and $\partial_t u$ by $\langle \Delta u, \phi \rangle = \langle u, \Delta_x \phi \rangle$ and $\langle \partial_t u, \phi \rangle = -\langle u, \partial_t \phi \rangle$. By a solution of (1.5)–(1.7) we mean $R \in \mathscr{S}_0'$ with

$$(\partial_t^2 - \Delta) R = \delta(x - x_0)\, \delta(t) \tag{2.2}$$

and

$$R = 0, \qquad \text{for} \quad t < 0. \tag{2.3}$$

Here we set $y = x_0$. Let us show, that R exists and is uniquely defined. By $\mathscr{S}_{0,x}$ we denote the space of all $\phi \in C^\infty(\bar{\Omega})$ with $\Delta^m \phi|_{\partial B} = 0$ for $m = 0, 1, 2, \ldots$, and $\sup_{x \in \bar{\Omega}} | x^\beta D^\alpha | < \infty$ for all multi-indices α, β.

Just as in the proof of Lemma 2.1 of [8] it follows that for every $\phi \in \mathscr{S}_{0,x}$ there exists $w(x, t) = U(t)\phi \in \mathscr{S}_{0,x}$ with

$$\begin{aligned} (\partial_t^2 - \Delta) w &= 0, \qquad \text{for} \quad t \leqslant 0 \\ w(x, 0) &= 0 \\ \partial_t w(x, 0) &= -\phi(x). \end{aligned}$$

Using imbedding theorems in Sobolev spaces, we also obtain

$$| w(x_0, t)| \leqslant C \sup_{|\alpha|\,|\beta| \leqslant 2} \sup_{x \in \bar{\Omega}} |x^\beta D^\alpha \phi|,$$

where the constant C is independent of t. These properties of w show that

$$v(x, t) = V(t)\phi = \int_t^\infty U(t-s)\,\phi(y, s)\,ds, \qquad \phi \in \mathscr{S}_0,$$

satisfies $(\partial_t^2 - \Delta)v = \phi$ and $|v(x_0, 0)| \leqslant C \sup_{|\alpha|\,|\beta| \leqslant 2} \sup_{z \in \bar{\Omega} \times \mathbb{R}} |z^\beta D^\alpha \phi|$. Moreover, we have $v(x_0, 0) = 0$ if $\operatorname{supp} \phi \subseteq \{t < 0\}$. The distribution $R \in \mathscr{S}'_0$ defined by

$$\langle R, \phi \rangle = \langle \delta(x - x_0)\,\delta(t), v(x, t) \rangle = v(x_0, 0)$$

is therefore a solution of (2.2), (2.3).

We show next that R is uniquely defined. Note first, that $x \to V(t)\phi \in \mathscr{S}_{0,x}$ for all $\phi \in \mathscr{S}_0$, which follows from a domain of dependence argument and the fact that ϕ is rapidely decreasing. This implies $\xi(t)\,V(t)\phi \in \mathscr{S}_0$, where $\xi \in C^\infty(\mathbb{R})$ satisfies

$$\begin{aligned} \xi(s) &= 0, \qquad s < -2, \\ &= 1, \qquad s > -1. \end{aligned}$$

Now suppose that S is another solution of (2.2), (2.3). Then we have $\langle R - S, (\partial_t^2 - \Delta)\phi \rangle = 0$ for all $\phi \in \mathscr{S}_0$, hence, in particular,

$$\begin{aligned} 0 &= \langle R - S, (\partial_t^2 - \Delta)(\xi V(t)\phi) \rangle \\ &= \langle R - S, (\partial_t^2 \xi)\,V(t)\phi + 2(\partial_t \xi)(\partial_t V(t)\phi) \rangle \\ &\quad + \langle R - S, \xi\phi \rangle. \end{aligned}$$

The first term on the right-hand side of this equation vanishes, since

$$\operatorname{supp}[(\partial_t^2 \xi)\,V(t)\phi + 2(\partial_t \xi)(\partial_t V(t)\phi)] \subseteq \{t < 0\},$$

and $R - S$ vanishes for $t < 0$. This yields $\langle R - S, \xi\phi \rangle = 0$, hence $\langle R - S, \phi \rangle = 0$, as $\operatorname{supp}(1 - \xi)\phi \subseteq \{t < 0\}$. Since ϕ is arbitrary, we conclude $R = S$.

In the next step of the proof we verify the following lemma.

LEMMA 2.1. *Let x_B be a regular point, and suppose that x_B is a point of reflection of a ray starting at x_0.*

(i) *Then there exists a relatively open neighborhood U of x_B in $\bar{\Omega}$ and $v \in C^\infty(U \times \mathbb{R})$ with*

$$R(x, x_0, t) = R_\infty(x, x_0, t) + v(x, x_0, t)$$

for $(x, t) \in U \times \mathbb{R}$. *Here,*

$$R_\infty(x, x_0, t) = (4\pi\,|x - x_0)^{-1}\delta(t - |x - x_0|) + \sum_{m=-1}^{\infty} \psi(\lambda_m(t - \Phi))\,H_m(t - \Phi)\,p_m, \qquad (2.4)$$

where the sequence $\lambda_m > 0$ *tends to infinity sufficiently fast.*

(ii) *If none of the rays starting at* x_0 *is reflected at* x_B, *then we have* $R \in C^\infty(U \times \mathbb{R})$.

(iii) *For every multi index* $\alpha \in \mathbb{Z}^4$ *and every* $r > 0$ *there exist constants* T, C, $C_1 > 0$ *such that*

$$\int_{\Omega_r} |D^\alpha R|^2\,dx \leqslant C_1 e^{-Ct}, \qquad t > T.$$

Here we set $\Omega_r = \{x \in \Omega \mid |x| \leqslant r\}$.

Proof. (i) Choose a bounded, relatively open neighborhood $V \subseteq \bar{\Omega}$ of x_B such that Φ is defined in V. The definitions (1.8) of ψ and H_m imply that the series in (2.4) is uniformly convergent in V if $\lambda_m \to \infty$ sufficiently fast. We show that we can choose λ_m such that

$$(\partial_t^2 - \Delta)\,R_\infty \in C^\infty(V \times \mathbb{R}). \qquad (2.5)$$

From the definition (1.8) of H_m and ψ it follows that

$$|D^\alpha(H_m(t - \Phi)\,p_m)| \leqslant C_m\,|t - \Phi|^{m - |\alpha|}, \qquad (2.6)$$

for (x, t) with $|t - \Phi(x)| \leqslant 1$, and

$$|D^\alpha \psi(\lambda_m(t - \Phi))| \leqslant C'_m \lambda_m^{|\alpha|}, \qquad \lambda_m \geqslant 1, \qquad (2.7)$$

uniformly in V, where $\alpha \in \mathbb{Z}^4$, $|\alpha| \leqslant m - 1$ in (2.6) Let

$$R_l = (4\pi\,|x - x_0|)^{-1}\delta(t - |x - x_0|) + \sum_{m=-1}^{l} \psi(\lambda_m(t - \Phi))\,H_m(t - \Phi)\,p_m.$$

By some calculations we obtain that

$$\langle R_l, (\partial_t^2 - \Delta)\phi\rangle = \langle f_l + g_l + h_l, \phi\rangle \qquad (2.8)$$

for $\phi \in \mathscr{S}_0$ with $\operatorname{supp} \phi \subseteq V \times \mathbb{R}$. Here

$$f_l = 2 \sum_{m=-1}^{l} \nabla'\psi \cdot \nabla'(H_m p_m) \in C^\infty,$$

$$g_l = \sum_{m=-1}^{l} H_m p_m(\partial_t^2 - \Delta)\psi \in C^\infty,$$

and

$$h_l = \sum_{m=-1}^{l} \psi(\partial_t^2 - \Delta)(H_m p_m),$$

where $\nabla' = (i\,\partial_{x_1},..., i\,\partial_{x_3}, \partial_t)$, and the arguments of ψ and H_m are the same as above. Note that $h_l \in C^{l-1}(V \times \mathbb{R})$, since (1.9), (1.10), and the fact that $\psi = 1$ near 0 imply

$$h_l = \psi(\lambda_l(t - \Phi))\, H_l(t - \Phi)\, \Delta p_l \in C^{l-1} \tag{2.9}$$

in a neighborhood of (x, t) with $t = \Phi(x)$.

Leibniz' rule, (2.6), and (2.7) together yield

$$\begin{aligned} |D^\alpha[\psi(\partial_t^2 - \Delta)(H_m p_m)]| &\leqslant C_m \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \lambda_m^{|\beta|}\, |t - \Phi|^{m-2+|\beta|-|\alpha|} \\ &\leqslant C'_m \sum_{\beta \leqslant \alpha} \lambda_m^{|\beta|} \lambda_m^{-m+2-|\beta|+|\alpha|} \\ &= C''_m \lambda_m^{2-m+|\alpha|} \leqslant 2^{-m}, \end{aligned}$$

for $|\alpha| < m - 2$ and λ_m sufficiently great. Here we used that $\psi(\lambda_m(t - \Phi)) = 0$ for $\lambda_m |t - \Phi| \geqslant 1$. This estimate and (2.9) show that h_l converges to an infinitely differentiable function for $l \to \infty$, and in the same way we can achieve this for f_l and g_l. On the other hand, $(\partial_t^2 - \Delta) R_l \to (\partial_t^2 - \Delta) R_\infty$, so (2.5) follows from (2.8).

We also have that R_∞ is infinitely differentiable away from the points (x, t) with $t = \Phi(x)$ if $\lambda_m \to \infty$ sufficiently fast. We omit the proof, since it is almost the same as the proof given above.

Let $v(x, t) = R(x, x_0, t) - R_\infty(x, x_0, t)$. For the proof of (i) it suffices to show that $v \in C^\infty(U \times \mathbb{R})$ for a suitable relatively open neighborhood $U \subseteq \bar{\Omega}$ of x_B. First we show that $(x_B, t) \notin \operatorname{sing\,supp} v$ for all t. Let $t_B = |x_B - x_0|$ and

$$\rho_0 = (x_0, 0, -\tfrac{1}{2}(x_B - x_0)\,|x_B - x_0|^{-1}, \tfrac{1}{2}) \in T^*_{(x_0,0)}(\Omega \times \mathbb{R}).$$

Let γ: $[0, t_B] \to (T^*(\Omega \times \mathbb{R})\backslash 0) \cup T^*(\partial\Omega \times \mathbb{R})$ be the integral curve of the broken Hamiltonian flow with $\gamma(0)=\rho_0$, so that $\gamma(t)\in T^*(\Omega\times\mathbb{R})$ for $0\leqslant t<t_B$ and $\gamma(t_B)\in T^*_{(x_B,t_B)}(\partial\Omega\times\mathbb{R})$. The results proved in [3, 5, 6, 9] show that if $\rho\in WF_b(R(\cdot,x_0,\cdot))$ has the x-coordinate equal to x_B, then its t coordinate is equal to t_B.

Note that we need assumption (A3) at this point, since the results in [6] have been proved under this assumption. As noted above, the same fact holds for R_∞, which implies that $(x_B,t)\notin\operatorname{sing\,supp} v$ for $t\neq t_B$. Hence it sufficies to show that $(x_B,t_B)\notin\operatorname{sing\,supp} v$.

Observe that $\operatorname{supp} v(\cdot,t)\subseteq\{|x-x_0|\leqslant t\}$ for $t<t_B$, and we shall see that

$$\Pi(\gamma(t))\notin\operatorname{supp} v(\cdot,t) \tag{2.10}$$

for $t<t_B$. Here $\Pi\colon T^*(\Omega\times\mathbb{R})\to\Omega\times\mathbb{R}$ is the natural projection. Therefore we have $\operatorname{dist}(\operatorname{supp} v(\cdot,t),x_B)>t_B-t$. This fact, $v|_{x\in\partial B}=0$, and $(\partial_t^2-\Delta)v\in C^\infty(V\times\mathbb{R})$ together imply that $(x_B,t_B)\notin\operatorname{sing\,supp} v$, which follows from the transversal case of the reflection of singularities theorem. Alternatively, it can be seen by standard energy estimates.

Equation (2.10) follows from a domain of dependence argument, which shows that in a neighborhood of $\Pi(\gamma(t))$ for $t<t_B$ the Riemann function R is equal to the free-space Riemann function defined in (1.11). By definition, this is also the case for R_∞, which proves (2.10), hence $(x_B,t)\notin\operatorname{sing\,supp} v$ for all t.

From this we conclude that for every bounded interval I there exists a neighbourhood U of x_B with $U\times I\cap\operatorname{sing\,supp} v=\varnothing$. Since the definition of R_∞ and the reflection of singularities theorem show that $V\cap\operatorname{sing\,supp} v(\cdot,t)=\varnothing$ for large t, the proof of (i) is complete.

(ii) This is a direct consequence of the propagation of singularities theorem.

(iii) Choose $r>0$ such that $B\subseteq\{|x|<r\}$. By the propagation of singularities theorem we know that there exists t_0 such that $\operatorname{sing\,supp} R(\cdot,t,x_0)\subseteq\{|x|>4r\}$ for all $t\geqslant t_0$. For such t we decompose R in the following way: Let $\xi_1,\xi_2\in C^\infty(\mathbb{R}^3)$ with

$$\begin{aligned}\xi_1(x)&=0, && |x|<3r,\\ &=1, && |x|>4r,\\ \xi_2(x)&=0, && |x|<r,\\ &=1, && |x|>2r.\end{aligned}$$

Let v_1 be the solution of

$$(\partial_t^2 - \Delta)\, v_1 = 0 \qquad \text{in } \Omega \times [t_0, \infty)$$

$$v_1|_{x \in \partial B} = 0$$

$$\partial_t^m v_1(x, t_0) = (1 - \xi_1(x))\, \partial_t^m R(x, x_0, t_0), \qquad m = 0, 1.$$

It follows from assumption (A1) and the results of [8] that

$$\int_{\Omega_r} |D^\alpha v_1|^2\, dx \leqslant C_1 e^{-Ct} \qquad \text{for suitable } C, C_1 > 0.$$

Let v_2 be the free space solution of

$$(\partial_t^2 - \Delta)\, v_2 = 0 \qquad \text{in } \mathbb{R}^3 \times [t_0, \infty),$$

$$\partial_t^m v_2(x, t_0) = \xi_1(x)\, \partial_t^m R(x, x_0, t_0), \qquad m = 0, 1.$$

The propagation of singularities theorem implies that v_2 has the same singularities as R, and Huygens' principle implies that there is t_1 such that $v_2(x, t) = 0$ for $|x| < 2r$ and $t \geqslant t_1$, as supp $\xi_1(\cdot)\, R(\cdot, x_0, t_0)$ is compact. The function $(\partial_t^2 - \Delta)(\xi_2(x)\, v_2)$ therefore is infinitely differentiable and has compact support in $\Omega \times (t_0, \infty)$. Hence, the solution v_3 of

$$(\partial_t^2 - \Delta)\, v_3 = -(\partial_t^2 - \Delta)(\xi_2 v_2) \qquad \text{in } \Omega \times \mathbb{R},$$

$$v_3|_{x \in \partial B} = 0,$$

$$v_3 = 0, \qquad \text{for} \quad t < t_0,$$

also satisfies $\int_{\Omega_r} |D^\alpha v_3|^2\, dx \leqslant C_1 e^{-Ct}$.

$v_1 + \xi_2 v_2 + v_3$ is a solution of the wave equation for $t > t_0$, vanishes at ∂B and is equal to R at $t = t_0$. Since also the time derivatives coincide at $t = t_0$, we have $R = v_1 + \xi_2 v_2 + v_3$. From the properties of v_1, v_2 and v_3 the assertion of (iii) thus follows.

In the last step of the proof we study the Fourier transform $\hat{R}$ of R and show that it is equal to G. As usual we define the partial Fourier transform $\hat{R}(x, x_0, k)$ of $R(x, x_0, t)$ with respect to t as follows: $\hat{R} \in \mathscr{S}_0'$ is the distribution defined by $\langle \hat{R}, \phi \rangle = \langle R, \hat{\phi} \rangle$ for all $\phi \in \mathscr{S}_0$, where

$$\hat{\phi}(x, t) = \int \phi(x, k)\, e^{ikt}\, dk.$$

Let $\mathscr{D}_{0,x}$ be the subset of $\mathscr{S}_{0,x}$ of all functions with bounded support, supplied with the suitably modified topology of Schwarz' space $\mathscr{D}$. For $\phi \in \mathscr{D}_{0,x}$ and $\chi \in \mathscr{S}(\mathbb{R})$ we obtain

$$\langle \hat{R}, \phi(x)\, \chi(k) \rangle = \langle R, \phi(x)\, \hat{\chi}(t) \rangle = \int \langle R, e^{ikt} \phi(x) \rangle\, \chi(k)\, dk, \qquad (2.11)$$

where $\langle R(x, x_0, t), e^{ikt}\phi(x)\rangle$ is defined in an obvious way using Lemma 2.1 (iii). This formula shows that, for every k, $\hat{R}(\cdot, x_0, k)$ defines a distribution on $\mathscr{D}_{0,x}$ defined by

$$\langle \hat{R}(x, x_0, k), \phi(x)\rangle_x = \langle R, e^{ikt}\phi\rangle, \tag{2.12}$$

and since, by (2.2),

$$\langle -(\Delta + k^2)\hat{R}, \phi(x)\,\chi(k)\rangle = \langle \delta(x - x_0)\,\delta(t), \phi(x)\,\hat{\chi}(t)\rangle = \phi(x_0)\int \chi\, dk,$$

(2.11) implies

$$-(\Delta + k^2)\,\hat{R}(\cdot, x_0, k) = \delta(x - x_0). \tag{2.13}$$

Standard elliptic regularity theorems now yield $\hat{R}(\cdot, x_0, k) \in C^\infty(\Omega\backslash\{x_0\})$. Also, we can derive a representation formula for $\hat{R}$:

Choose $r > |x_0|$ such that $\bar{B} \subseteq \{|x| < r\}$, and choose $\xi \in C^\infty(\mathbb{R}^3)$ with

$$\begin{aligned} \xi(x) &= 0, \qquad |x| < r, \\ &= 1, \qquad |x| > r + 1. \end{aligned}$$

$\xi(x)R$ vanishes for $t < 0$ and is the free space solution of $(\partial_t^2 - \Delta)w = f$, where $f = -2\nabla\xi \cdot \nabla R - R\,\Delta\xi$. Therefore we have $\xi R = \tilde{R} * f$, where $\tilde{R}$ is the free space Riemann function defined in (1.11), and convolution is defined by

$$\langle \tilde{R} * f, \phi\rangle = \langle f(y, s), \langle \tilde{R}(y, x, t), \phi(x, t + s)\rangle\rangle, \qquad \phi \in \mathscr{S}(\mathbb{R}^4).$$

The function $(y, s) \mapsto \langle \tilde{R}(y, x, t), \phi(x, t + s)\rangle$ is not necessarily contained in $\mathscr{S}(\mathbb{R}^4)$, but it is rapidly decreasing with respect to s, hence, since $\operatorname{supp} f(\cdot, s)$ is bounded independently of s, the definition makes sense. Equations (1.11) and (2.12) now imply for $\phi \in \mathscr{D}_{0,x}$ with $\operatorname{supp}\phi \subseteq \{|x| > r + 1\}$:

$$\begin{aligned} \langle \hat{R}, \phi\rangle_x &= \langle \tilde{R} * f, e^{ikt}\phi\rangle = \langle f(y, s), e^{iks}\langle \tilde{R}(x, y, t), e^{ikt}\phi(x)\rangle\rangle \\ &= \langle \hat{f}(y, k), \int (4\pi\,|x - y|)^{-1}\, e^{ikt|x-y|}\,\phi(x)\,dx\rangle_y \\ &= \int \langle \hat{f}, (4\pi\,|x - y|)^{-1}\, e^{ik|x-y|}\rangle_y\,\phi(x)\,dx. \end{aligned}$$

Here we used the definition of f and the fact that $\operatorname{supp}\hat{f} \cap \operatorname{supp}\phi = \varnothing$. Thus we have for $|x| > r + 1$

$$\hat{R}(x, x_0, k) = \langle \hat{f}, (4\pi\,|x - y|)^{-1}\, e^{ik|x-y|}\rangle_y, \tag{2.14}$$

which is the desired representation formula.

Now we are ready to show that $\hat{R} = G$. Note first that $G(\cdot, x_0, k) \in \mathscr{D}'_{0,x}$ satisfies

$$-(\Delta + k^2)\, G(x, x_0, k) = \delta(x - x_0) \tag{2.15}$$

and

$$G(x, x_0, k) = \int (4\pi\, |x - y|)^{-1}\, e^{ik|x-y|}\, g(y, k)\, dy \tag{2.16}$$

for $|x| > r + 1$, where $g = -2\nabla\xi \cdot \nabla G - G\,\Delta\xi$. It suffices to show that $\langle \hat{R} - G, \phi \rangle = 0$ for all $\phi \in \mathscr{D}_{0,x}$. Let $w_r = -(G * \phi) \cdot \xi_1(|x| - r)$, where $\xi_1 \in C^\infty(\mathbb{R})$ with

$$\begin{aligned} \xi(s) &= 1, \qquad s < 0, \\ &= 0, \qquad s > 1. \end{aligned}$$

Then we have $w_r \in \mathscr{D}_{0,x}$ and, by (2.13) and (2.15),

$$\begin{aligned} 0 = \langle \hat{R} - G, (\Delta + k^2)\, w_r \rangle &= \lim_{r \to \infty} \langle \hat{R} - G, (\Delta + k^2)\, w_r \rangle \\ &= \langle \hat{R} - G, \phi \rangle - \lim_{r \to \infty} \int (\hat{R} - G)[2\nabla(G * \phi) \cdot \nabla\xi_1(|x| - r) \\ &\quad + (G * \phi)\, \Delta\xi_1(|x| - r)]\, dx. \end{aligned}$$

Partial integration and the asymptotic behavior of $\hat{R}$ and G, which can be obtained from (2.14) and (2.16), show that the last term of this equation is equal to

$$-\lim_{r \to \infty} \int [(\hat{R} - G)\, \nabla(G * \phi) - (G * \phi)\, \nabla(\hat{R} - G)] \cdot \nabla\xi_1(|x| - r)\, dx = 0,$$

hence $\langle \hat{R} - G, \phi \rangle = 0$ and therefore also $\hat{R} = G$, since ϕ is arbitrary.

It remains to calculate the Fourier transform in a neighborhood of x_B. Using (2.12), Lemma 2.1 (i), and the fact that $\operatorname{supp} R_\infty(x, x_0, \cdot)$ is bounded uniformly in U, we obtain

$$\hat{R}(x, x_0, k) = \langle R_\infty(x, x_0, t), e^{itk} \rangle_t + \hat{v}(x, t).$$

As $v \in C^\infty(U \times \mathbb{R})$ and v together with all derivatives decreases exponentially as $t \to \infty$, uniformly in U, we obtain $D_x^\alpha \hat{v} \to 0$ for $t \to \infty$, faster than any power of k, and uniformly in U. By construction the series in (2.4) is uniformly convergent in U after termwise differentiation to arbitrary order. This yields

$$\begin{aligned} D_x^\alpha \langle R_\infty, e^{itk} \rangle_t &= D_x^\alpha (4\pi\, |x - x_0|)^{-1}\, e^{ik|x-x_0|} \\ &\quad + \sum_{m=-1}^{\infty} D_x^\alpha \langle \psi(\lambda_m(t - \Phi))\, H_m(t - \Phi)\, p_m, e^{itk} \rangle_t, \end{aligned}$$

from which the assertion of Theorem 1.1 follows by a simple computation.

3. Proof of Theorem 1.2

We use the notations introduced before Theorem 1.2. Let $V \subseteq \partial B_1 \cap \partial B_2$ be a neighborhood of x_B. For $x \in \mathbb{R}^3$ let

$$\varepsilon(x) = \inf\{|x-y| + |y-x_0| \mid y \in (\partial B_1 \cup \partial B_2)\backslash V\} - |x-x_0|.$$

Assumption (A2) implies that $\varepsilon_1 = \varepsilon(x_B) > 0$. We choose a sphere S with centre at x_B and radius $r < \varepsilon_1/4$. Then we have

$$\varepsilon(x) > \varepsilon_1/2 \tag{3.1}$$

for all $x \in S$. Assumption (A2) also implies, that the surfaces $\{\Phi_1(x) = t\} \cup \{|x-x_0| = t\}$ and $\{\Phi_2(x) = t\} \cup \{|x-x_0| = t\}$ have positive distance from x_B for $t > |x_B - x_0|$. Φ_1 and Φ_2 are the phase functions to B_1 and B_2. In particular, there exists $\varepsilon_2 > 0$ such that the distances of these surfaces from x_B are greater than ε_2 for all $t > |x_B - x_0| + \varepsilon_1/2$. We choose $r < \varepsilon_2$. Thus the sphere S does not intersect these surfaces for $t \geqslant |x_B - x_0| + \varepsilon_1/2$.

Let $R^{(1)} = \tilde{R} + v_1$ and $R^{(2)} = \tilde{R} + v_2$ be the Riemann functions for B_1 and B_2, respectively. Using (3.1) we obtain from a domain of dependence argument, that v_1 and v_2 depend in $S \cap \bar{\Omega}$ for $t \leqslant |x_B - x_0| + \varepsilon_1/2$ only on the values of $\tilde{R}|_V$, hence $v_1 = v_2$ in $S \cap \bar{\Omega}$ for $t \leqslant |x_B - x_0| + \varepsilon_1/2$.

As in the proof of Theorem 1.1 we see that $\operatorname{sing\,supp} R^{(1)} \subseteq \{\Phi_1 = t\} \cup \{|x-x_0| = t\}$ and $\operatorname{sing\,supp} R^{(2)} \subseteq \{\Phi_2 = t\} \cup \{|x-x_0| = t\}$. Therefore S does not intersect $\operatorname{sing\,supp} R^{(1)}$ and $\operatorname{sing\,supp} R^{(2)}$ for $t \geqslant |x_B - x_0| + \varepsilon_1/2$, hence $R^{(1)}$ and $R^{(2)}$ are infinitely differentiable in $S \cap \bar{\Omega}$ for $t \geqslant |x_B - x_0| + \varepsilon_1/2$. Summing up, we obtain $R^{(1)} - R^{(2)} \in C^\infty(S \cap \bar{\Omega} \times \mathbb{R})$. Fourier transformation of $R^{(1)}$ and $R^{(2)}$ now proves the assertion of Theorem 1.2.

4. Proof of Lemma 1.3

The proof is straightforward. Just as in the proof of Theorem 1.1 it can be shown that the Fourier transform of the solution v of (1.13)–(1.15) is a solution of (1.1)–(1.3). The results mentioned before Lemma 1.3 show that v is infinitely differentiable at $x_0 \notin M(\mu)$. Hence the statement of the lemma follows from the properties of the Fourier transformation.

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